Modified Adomian Decomposition Method for Solving Volterra Integro-Differential Equations

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Abstract This research article effectively demonstrates the implementation of the modified Adomian decomposition method (MADM). Using a numerical procedure called MADM, some classes of Volterra integro differential equations can be solved can be solved with easily computational and high degree of acuracy. The procedure relies on ADM approximate series solutions, Laplace transform, and Pade approximants. The efficacy and dependability of MADM is tested through a numerical example. The results acquired reveal that the provided approach is highly effective and robust in addressing this differential equation.

Keywords HPM procedure · Laplace transform · Pade approximants · Integral equations

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1 Introduction

Volterra integro-differential equations are commonly found in various scientific areas such as astronomy, biology, biotechnology, engineering, physics, and radiology. There are numerous applications for these equations, including diffusion processes, and cell growth. The concept of Volterra integro-differential equations was initially introduced by Volterra himself when he encountered a situation where a single equation consisted of both differential and integral operators. He was exploring a model for population growth to investigate the influence of heredity. As a result. This novel form of equation is referred to as the Volterra integro-differential equation. These equations can be expressed as:

$$u^{(n)} = f(x) + \lambda \int_0^x k(x, t)u(t)dt. \tag{1}$$

It is crucial to establish and delineate the initial conditions.

$$u(0),\ u'(0),\ u''(0), \ldots,\ u^{(n-1)}(0).$$

Given the widespread use of differential equations in the modeling and characterization of real-world phenomena, the solutions to these equations are of significant significance in the realms of applied mathematics and engineering. However, solving these equations can be challenging, especially when dealing with strongly nonlinear equations. Therefore, it is necessary to obtain exact solutions or accurate approximations with a high level of precision. The behavior of the phenomena can be studied and understood through these solutions. To achieve this goal, various numerical and approximation methods have been employed and developed for solving a wide range of equations [1–29]. The Adomian decomposition method is widely recognized as a systematic technique used to solve a wide range of linear and nonlinear equations. These equations can include ordinary differential equations, partial differential equations, integral equations, integro-differential equations, and other related equations.

The aim of this work is to improve the ADM to provide accurate solutions for Volterra integro-differential equations. To accomplish this, an alternative technique called the MADM will be utilized. MADM alters the series solution for a specific category of Volterra integro-differential equations by incorporating the Laplace transformation on the truncated series derived from ADM. Consequently, the transformed series is converted into a meromorphic function through the utilization of Padé approximants. Ultimately, the inverse Laplace transform is applied to the resultant analytic solution, generating the exact solution.

Section 2 provides a concise exposition of the fundamental principles underlying the ADM [30–32], Padé approximation, and Laplace transformation. In Sect. 3, we present concrete instances to exemplify the applicability and efficacy of our proposed methodology. Finally, Sect. 4 offers a comprehensive conclusion that encapsulates the key discoveries and contributions of this research endeavor.

2 Research Methodology

2.1 ADM Procedure

The essence of the ADM lies in its focus on differential equations of the following form:

$$Lu + Ru + Nu = g, \tag{2}$$

here g and u are the systems input, and output, L and N is the linear and nonlinear operators, R is the linear remainder operator is the operator. We note that the selection of the linear operator is intended to yield an easily invertible operator, leading to straightforward integrations. Furthermore, we emphasize the significance of the choice for L. and its inverse L^{-1} are decided by the particular equation to be solved (Adomian). Moreover, we choose $L = \frac{d^m}{dx^m}(.)$ for mth-order differential equation and thus its inverse L^{-1} follows as m-fold definite integration operator from x_0 to x. We have $L^{-1}Lu = u - \psi$, where ψ incorporates the initial values as

$$\psi = \sum_{v=0}^{m-1} \beta_v \frac{(x - x_0)^v}{v!}.$$

Applying L^{-1} to both sides of Eq. (2) yields:

$$u = g(x) - L^{-1}[Ru + Nu], \tag{3}$$

where $g(x) = \psi + L^{-1}g$. The ADM solution:

$$u(x) = \sum_0^{\infty} u_n, \tag{4}$$

And the nonlinear term Nu:

$$Nu = \sum_0^{\infty} A_n, \tag{5}$$

Knowing that, the Adomian polynomials A_n, depending upon $u_0, u_1, \ldots, u_n$, that are gained form the nonlinearity formula

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[f\sum_{n=0}^{\infty}\lambda^n u_n\right]_{\lambda=0} \qquad n = 0, 1, 2, \ldots \tag{6}$$

where λ is a parameter.

3 Numerical Test

In this section of the discussion, we will analyze one illustration. This example is taken into consideration and is illustrative of the technique for Volterra integro-differential equations.

Example 3.1 Consider the provided Volterra integral differential equation.

$$u''(x) = x + \int_0^x (x-t)u(t)dt \quad u(0) = 1,\ u'(0) = 1 \tag{7}$$

To apply the MADM procedure for this problem, first we employ the ADM by integrating Eq. (7), this yields to

$$u(x) = x + \frac{x^3}{6} + L^{-1}\left(\int_0^x (x-t)u(t)dt\right). \tag{8}$$

According to the ADM process, we possess the following

$$u_0(x) = x + \frac{x^3}{6}, \tag{9}$$

and

$$u_{n+1}(x) = L^{-1}\left(\int_0^x u_n(t)dt\right). \tag{10}$$

Which yields to

$$u_1(x) = L^{-1}\left(\int_0^x (x-t)u_0(t)dt\right) = \frac{x^5}{5!} + \frac{x^7}{7!}, \tag{11}$$

$$u_2(x) = L^{-1}\left(\int_0^x (x-t)u_1(t)dt\right) = \frac{x^9}{9!} + \frac{x^{11}}{11!}, \tag{12}$$

As a result, the series-form solution to Eq. (7) provided by

$$u(x) = x + \frac{x^3}{6} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \frac{x^{11}}{11!}, \tag{13}$$

And in the limit of an infinite number of terms, the sequence converges to the exact solution $u(x) = sinhx$. Numerical results are displayed in Table 1 and are presented graphically in Fig. 1. To improve the accuracy, we employ the Laplace transform on Eq. (13) as follows.

$$L(u(t)) = \frac{1}{s^{12}} + \frac{1}{s^{10}} + \frac{1}{s^8} + \frac{1}{s^6} + \frac{1}{s^4} + \frac{1}{s^2}, \tag{14}$$

Assum $= \frac{1}{z}$, then

$$L(u(t)) = z^2 + z^4 + z^6 + z^8 + z^{10} + z^{12}, \tag{15}$$

Then we apply Pade approximate of order $\left[\frac{3}{3}\right]$ on (15), yields to $\left[\frac{3}{3}\right] = \frac{z^2}{1-z^2}$, Recalling $z = \frac{1}{s}$, gives $\left[\frac{3}{3}\right] = \frac{1}{(1-\frac{1}{s^2})s^2}$.

Using the Padé approximation and the inverse Laplace transform, one obtains $u(x) = sinhx$.

Table 1 Numerical results

x	Exact solution	ADM solution	Absolute error
0	1	1	0
0.2	0.2013360025	0.2013360025	0
0.4	0.4107523258	0.4107523258	1.05×10^{-15}
0.6	0.6366535821	0.6366535821	2.10×10^{-13}
0.8	0.8881059822	0.8881059822	8.86×10^{-12}
1	1.1752011936	1.1752011935	1.61×10^{-10}

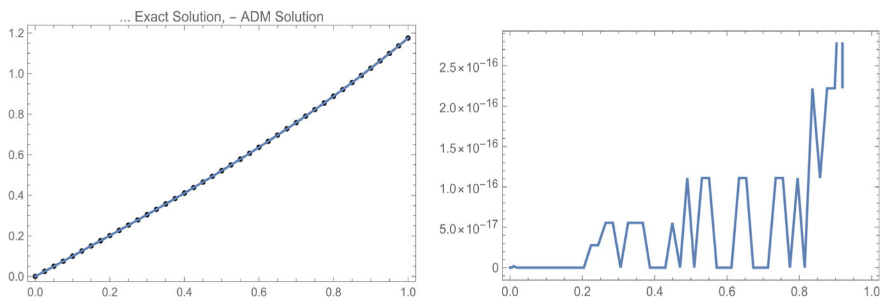

Fig. 1 **a** Exact and approximate solutions, **b** the absolute error

4 Conclusion

The MADM is efficiently used in this research paper to obtain precise solutions for Volterra integro-differential equations. In this case, an analysis is conducted to assess the applicability and effectiveness of this technique. The obtained outcomes are presented as exact solutions, highlighting the significant power of the MADM.

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